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ABSTRACT

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Analysis of the stressed state of a thin infinite plate with a nearly circular hole having a time-variable pressure applied to the edges of the hole. A solution is obtained using a modification of the "perturbed boundary shape" method, proposed by Savin and Guz' for the solution of static problems of stress concentration near noncircular holes in elastic shells. Using this method, the problem is reduced to the solution of two Helmholtz equations. It is shown that at certain frequencies, the stress concentration is higher by 15 to 20 percent than in the static case.

Author

A solution is presented for the state of stress in a thin infinite plate with a hole whose contour is subject to a pressure which varies with time according to the law $e^{-i\omega t}$. The contour Γ of the hole is assumed to be close to a circular one in the sense that the function $z = \omega(\zeta)$ which carries out the conformal transformation of the infinite plane with the hole consisting of a unit circle, on an infinite plane z with hole Γ, has the form /134*

$$\omega(\zeta) = a_0(\zeta + \varepsilon\zeta^{-N}), \qquad (1)$$

* Numbers given in the margin indicate the pagination in the original foreign text.

where $z = re^{i\theta}$; $\zeta = \rho e^{i\gamma}$; a_0 is the radius of the circular hole closely approximated by the contour Γ.

By using appropriate values of N and ε we can obtain an elliptic, triangular or square hole (ref. 2). The perturbed boundary shape method (ref. 1) is used in the form proposed in reference 3 for solving static problems of stress concentration near noncircular holes in elastic shells.

We introduce the dimensionless quantities

$$\begin{aligned} &\bar{r} = \frac{r}{a_0}; \quad \bar{t} = \frac{c_2 t}{a_0}; \quad \bar{u} = \frac{Eu}{2(1+\nu)a_0\sigma_0}; \quad \bar{v} = \frac{Ev}{2(1+\nu)a_0\sigma_0}; \\ &\bar{\sigma}_n = \frac{\sigma_n}{\sigma_0}; \quad \bar{\sigma}_s = \frac{\sigma_s}{\sigma_0}; \quad \bar{\tau}_{ns} = \frac{\tau_{ns}}{\sigma_0}; \quad c_2 = \sqrt{\frac{G}{\mu}}; \quad \bar{\omega} = \frac{a_0\omega}{c_2}, \end{aligned} \tag{2}$$

where r is the radial coordinate; t is the time; u and v are the radial and tangential displacements; E, G and ν are respectively Young's modulus, the Shear modulus and the Poisson ratio; c_2 is the velocity of transverse waves; σ_0 is the amplitude of the pressure applied to the contour; σ_n, σ_s, τ_{ns} are components of the state of stress; μ is the density; ω is the angular frequency.

The problem is reduced to two Helmholtz equations

$$\nabla^2\varphi + \xi^2\omega^2\varphi = 0; \qquad \nabla^2\psi + \omega^2\psi = 0, \tag{3}$$

which are supplemented with the boundary conditions

$$\sigma_n|_\Gamma = -1; \qquad \tau_{ns}|_\Gamma = 0. \tag{4}$$

Here φ and ψ are respectively the potentials of the longitudinal and transverse waves associated with the displacement vector by means of the relationship

$$\vec{U}(u,v) = \vec{\nabla}\varphi + \vec{\nabla} \times \vec{k}\psi.$$

where $\vec{k}$ is the opt of the normal to the plane of the plate; $\varepsilon^2 = \frac{1-\nu}{2}$.

The conditions of radiation must be added to conditions (4).

Below we use only the symbols introduced in equations (2), and the bars over the letters are not used. By using equation (1) for an elliptic hole we obtain the following

$$N = 1; \quad \varepsilon = \frac{a-b}{a+b}; \quad r = \varrho\sqrt{1 + 2\varepsilon\varrho^{-2}\cos 2\gamma + \varepsilon^2\varrho^{-4}};$$
$$\theta = \operatorname{arctg}\left(\frac{\varrho - \varepsilon\varrho^{-1}}{\varrho + \varepsilon\varrho^{-1}}\operatorname{tg}\gamma\right); \quad e^{i\alpha} = \frac{\overline{\omega(\zeta)\zeta\omega'(\zeta)}}{|\omega(\zeta)||\zeta||\omega'(\zeta)|},$$

where α is the angle between the radial direction and the normal to the lines on the plane z which are obtained by transforming the lines ρ = const in the plane ζ.

We expand r, θ, $e^{i\alpha}$ into power series of ε. If Φ designates φ or Ψ we can obtain the following expansion

$$\Phi(r,\theta) = \Phi(\varrho,\gamma) + \varepsilon\left(\frac{\cos 2\gamma}{\varrho}\frac{\partial}{\partial\varrho} - \frac{\sin 2\gamma}{\varrho^2}\frac{\partial}{\partial\gamma}\right)\Phi(\varrho,\gamma)$$
$$+\varepsilon^2\left(\frac{1-\cos 4\gamma}{4\varrho^3}\frac{\partial}{\partial\varrho} + \frac{\sin 4\gamma}{2\varrho^4}\frac{\partial}{\partial\gamma} + \frac{1+\cos 4\gamma}{4\varrho^2}\frac{\partial^2}{\partial\varrho^2}\right.$$
$$\left.-\frac{\sin 4\gamma}{2\varrho^3}\frac{\partial^2}{\partial\varrho\partial\gamma} + \frac{1-\cos 4\gamma}{4\varrho^4}\frac{\partial^2}{\partial\gamma^2}\right)\Phi(\varrho,\gamma).$$

We represent the stress components and the functions $\varphi(r,\theta)$, $\Psi(r,\theta)$ as power series of ε /135

$$\varphi(r,\theta) = \sum_{l=0}^{\infty}\varepsilon^l\varphi_l(r,\theta); \quad \psi(r,\theta) = \sum_{l=0}^{\infty}\varepsilon^l\psi_l(r,\theta); \qquad (5)$$
$$\sigma_n(r,\theta) = \sum_{l=0}^{\infty}\varepsilon^l\sigma_n^{(l)}(r,\theta);\ \ldots$$

To determine σ_n, σ_s, τ_{ns} in terms of stress components given by the polar coordinates we use the equations

$$
\begin{aligned}
\sigma_n &= \sigma_r \cos^2\alpha + \sigma_\theta \sin^2\alpha + 2\tau_{r\theta} \sin\alpha\cos\alpha;\\
\sigma_s &= \sigma_r \sin^2\alpha + \sigma_\theta \cos^2\alpha - 2\tau_{r\theta} \sin\alpha\cos\alpha;\\
\tau_{ns} &= (\sigma_\theta - \sigma_r) \sin\alpha\cos\alpha + \tau_{r\theta} (\cos^2\alpha - \sin^2\alpha).
\end{aligned} \tag{6}
$$

In these equations

$$
\begin{aligned}
\sigma_r &= -\nu\omega^2\varphi + 2\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\psi}{\partial\theta}\right);\\
\sigma_\theta &= -\omega^2\varphi - 2\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial^2\psi}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\psi}{\partial\theta}\right);\\
\tau_{r\theta} &= -\omega^2\psi + 2\left(\frac{1}{r}\frac{\partial^2\varphi}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\varphi}{\partial\theta} - \frac{\partial^2\psi}{\partial r^2}\right).
\end{aligned} \tag{7}
$$

By substituting the expansions into (3), (7) and (6) and collecting the coefficients for powers ε we obtain equations for the j th appoximation

$$
\nabla^2\varphi_j + \xi^2\omega^2\varphi_j = 0; \qquad \nabla^2\psi_j + \omega^2\psi_j = 0 \tag{8}
$$

whose solutions are

$$
\begin{aligned}
\varphi_j &= \sum_{n=0}^{\infty} [A_n^{(j)} H_n^{(1)}(\omega\xi r)\cos n\theta + B_n^{(j)} H_n^{(1)}(\omega\xi r)\sin n\theta];\\
\psi_j &= \sum_{n=0}^{\infty} [C_n^{(j)} H_n^{(1)}(\omega r)\cos n\theta + D_n^{(j)} H_n^{(1)}(\omega r)\sin n\theta],
\end{aligned}
$$

where $H_n^{(i)}$ is the Hankel function of the I kind.

The boundary conditions have the form

$$
\sigma_n^{(j)}|_\Gamma = f_1(\gamma); \qquad \tau_{ns}^{(j)}|_\Gamma = f_2(\gamma). \tag{9}
$$

In equations (9) the right part is determined from the preceding approximations while the operators containing φ_j, ψ_j in the left part coincide with the /136 corresponding operators used to determine stresses using the polar coordinates ρ, γ.

As a result of this the expressions giving the stresses for the problem formulated above have the following form (with an accuracy up to ε^3):

$$
\begin{aligned}
\sigma_n(r,\theta) &= e^{-i\omega t}\{\sigma_n^{(0)}(\varrho,\omega) + \varepsilon\sigma_n^{(1)}(\varrho,\omega)\cos 2\gamma + \varepsilon^2[\sigma_{n0}^{(2)}(\varrho,\omega) + \sigma_n^{(2)}(\varrho,\omega)\cos 4\gamma]\};\\
\tau_{ns}(r,\theta) &= e^{-i\omega t}[\varepsilon\tau^{(1)}(\varrho,\omega)\sin 2\gamma + \varepsilon^2\tau^{(2)}(\varrho,\omega)\sin 4\gamma];\\
\sigma_s(r,\theta) &= e^{-i\omega t}\{\sigma_s^{(0)}(\varrho,\omega) + \varepsilon\sigma_s^{(1)}(\varrho,\omega)\cos 2\gamma + \varepsilon^2[\sigma_{s0}^{(2)}(\varrho,\omega) + \sigma_s^{(2)}(\varrho,\omega)\cos 4\gamma]\}.
\end{aligned}
\qquad (10)
$$

The real parts R in (10) give the stresses when t = 0 while the imaginary parts I give them when $t = T/4$, $T = 2\pi/\omega$. The absolute values $\sqrt{R^2 + I^2}$ give the maximum stresses.

If in (10) we let ω approach zero and use the asymptotic properties of cylindrical functions we obtain, in the limit, expressions which coincide with the power expansion of the stresses for the corresponding static problem. This situation makes it possible for us to establish the convergence of the solution which is obtained. Thus for the example cited below the error in determining σ_s along the contour of the hole when $\omega \to 0$ is less than 1.5 percent.

Numerical results were obtained for $\varepsilon = 1/7$ which corresponds to $a/b = 4/3$. Calculations were carried out for $\nu = 0.28$. Table 1 presents the values of $\sigma_{n_{max}}$ as a function of ρ for $\omega = 1, 2, 3$. We can see from table 2 the variation in σ_s along the hole contour. The corresponding values for r and θ are determined by means of equation (5).

The drawing represents the variation in the maximum stress ζ along the contour of the hole as a function of the frequency when $\theta = 0, \pi/2$. In the static case we have:

for $\varepsilon = 0$ (circular hole) $\quad \sigma_s = 1.0$;

for $\varepsilon = 1/7 \quad \theta = \pi/2 \quad \sigma_s = 0.35$;

for $\varepsilon = 1/7 \quad \theta = 0 \quad \sigma_s \doteq 1.65$.

The average curve is constructed for a circular hole while the upper and lower curves are for an elliptic hole with $\theta = \pi/2$ and $\theta = 0$, respectively. /137

TABLE 1

ρ	$\omega = 1$ (a) $\sigma_{n_{max}}$	(b) $\sigma_{n_{max}}$	$\omega = 2$ (a) $\sigma_{n_{max}}$	(b) $\sigma_{n_{max}}$	$\omega = 3$ (a) $\sigma_{n_{max}}$	(b) $\sigma_{n_{max}}$
1,0	1,00	1,00	1,00	1,00	1,00	1,00
1,2	0,79	0,72	0,91	0,87	0,92	0,89
1,4	0,68	0,61	0,84	0,80	0,87	0,81
1,6	0,61	0,55	0,81	0,76	0,84	0,78
1,8	0,56	0,52	0,75	0,71	0,78	0,70
2,0	0,53	0,49	0,71	0,68	0,75	0,66
2,5	0,45	0,45	0,64	0,62	0,67	0,58
3,0	0,43	0,42	0,59	0,57	0,62	0,53
3,5	0,40	0,36	0,55	0,54	0,57	0,49
4,0	0,38	0,32	0,51	0,49	0,54	0,43

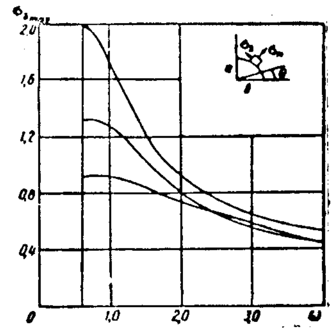

TABLE 2

		$\gamma = \frac{\pi}{12}$	$\gamma = \frac{\pi}{6}$	$\gamma = \frac{\pi}{4}$	$\gamma = \frac{\pi}{3}$	$\gamma = \frac{5\pi}{12}$	hole
$\omega=1$	$\mathrm{Re}\,\sigma_s^{(2)}$	0,991	0,914	0,848	0,783	0,764	0,894
	$\mathrm{Im}\,\sigma_s^{(2)}$	1,322	1,117	0,894	0,670	0,549	0,920
$\omega=2$	$\mathrm{Re}\,\sigma_s^{(2)}$	0,071	0,002	−0,014	0,000	0,015	0,009
	$\mathrm{Im}\,\sigma_s^{(2)}$	0,898	0,816	0,767	0,717	0,726	0,816
$\omega=3$	$\mathrm{Re}\,\sigma_s^{(2)}$	−0,057	−0,082	−0,017	−0,212	−0,283	−0,181
	$\mathrm{Im}\,\sigma_s^{(2)}$	0,613	0,539	0,507	0,475	0,500	0,544
$\omega=4$	$\mathrm{Re}\,\sigma_s^{(2)}$	−0,114	−0,160	−0,224	−0,287	−0,333	−0,232
	$\mathrm{Im}\,\sigma_s^{(2)}$	0,524	0,469	0,403	0,336	0,294	0,400

Thus, as we can see from the drawing for some values of the frequency the stress configuration is greater by 15-20 percent than for the static case.

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